Detecting Changes of Steady States Using the Mathematical Theory of Evidence

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The detection of changes in steady states is important in quality control, data reconciliation, and process monitoring applications. In a previous paper (Narasimhan et al., 1986) a composite statistical test was developed and evaluated for this purpose. In this note we present and evaluate an alternative method based on the mathematical theory of evidence developed by Shafer (1976). A key element of this approach is the assignment of beliefs to the different propositions of interest. This step usually involves subjective judgment. In this note we propose to make it less subjective by using the probability distribution of the measurements and certain limiting conditions. Simulation studies were carried out to compare the performance of this method with the multivariate statistical test developed by Narasimhan et al. (1986). The results show that both methods give the same performance. Thus this approach is an attractive alternative for detecting changes of steady states when the variables are independent. The belief function proposed here may also be used in other applications such as fault diagnosis (Kramer, 1987).

Model and Assumptions

The most important assumptions made are stated below. Other assumptions are the same as those used in Narasimhan et al. (1986).

1. A process undergoes a change in its steady state if the true values of one or more of its variables change.
2. Measurements of process variables contain only random errors, which are normally distributed with mean $\underline{0}$ and covariance matrix $\underline{Q}$.
3. A time period consists of N successive measurements. The process state can change from one period to another, but within each period the process is assumed to be in a steady state.
4. Successive measurements are mutually independent.
5. $\underline{Q}$ is unknown. It is diagonal and constant from one period to another.

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With the above assumptions, the measurement model is given by

$$\underline{x}_{kj} = \underline{x}_k + \underline{v}_{kj}, \quad j = 1, 2, \ldots N \tag{1}$$

$$\underline{v}_{kj} \sim N(\underline{0}, \underline{Q}), \quad j = 1, 2, \ldots N \tag{2}$$

where $\underline{x}_{kj}$: $p \times 1$ is the jth measurement vector in period k; $\underline{x}_k$: $p \times 1$ is a vector of true values of variables in period k, and $\underline{v}_{kj}$: $p \times 1$ is a vector of random measurement errors.

We are interested in detecting changes in the process state from one period to the next. We will therefore apply the test procedure to pairs of consecutive time periods as described in Narasimhan et al. (1986).

The Mathematical Theory of Evidence Method

We will now describe the mathematical theory of evidence (MTE) approach as applied to the detection of changes in steady states. A detailed description of the theory may be found in Shafer (1976).

Let us consider two successive periods, 1 and 2. We are interested in the following three propositions regarding the state of the process in these periods:

1. The process is in a steady state
2. The process state has changed from period 1 to 2
3. We are not certain whether the process state has changed or not

In the MTE approach, a fraction of a unit of belief is ascribed to each of the possible propositions, such that the total belief is equal to unity. The function that ascribes the belief is known as the belief function. The measurement for each variable is used in assigning beliefs for the three propositions. The beliefs from all variables are then combined to arrive at a decision. Thus in order to apply the MTE, we should choose a belief function and a rule which can be used to combine the beliefs that are derived from each variable.

Belief function

Let a statistic for each variable i be defined as

$$t_{1,i}^2 = N(\bar{x}_{2,i} - \bar{x}_{1,i})^2/(s_{1,i} + s_{2,i}) \quad (3)$$

where $\bar{x}_{k,i}$ is the mean value of the measurements for variable i in period k,

$$\bar{x}_{k,i} = \left(\sum_{j=1}^{N} x_{kj,i}\right)\bigg/N; \quad k = 1, 2 \quad (4)$$

and $s_{k,i}$ is the sample variance of variable i in period k, which is given by

$$s_{k,i} = \sum_{j=1}^{N} (x_{kj,i} - \bar{x}_{k,i})^2/(N-1); \quad k = 1, 2 \quad (5)$$

It can be shown that $t_{1,i}^2$ is a random variable obeying Hotelling's T^2 distribution with numerator degrees of freedom 1, and denominator degrees of freedom $2N - 2$ (Narasimhan et al., 1986). Henceforth, unless otherwise stated, these default degrees of freedom are assumed when we use the notation T^2.

Let α be a chosen level of significance and $T^2(\alpha)$ be the upper α quantile of the T^2 distribution. Let $m_i(S)$, $m_i(NS)$, and $m_i(U)$ be the beliefs attributed to the propositions of steady state, nonsteady state, and uncertainty, respectively. We propose a belief function by which the beliefs are calculated purely from the T^2 statistical distribution according to the following formulas for every variable i.

Case A: $t_{1,i}^2 \leq T^2(\alpha)$

$$m_i(S) = Pr\{T^2 \geq t_{1,i}^2\} \quad (6)$$

$$m_i(NS) = 0 \quad (7)$$

$$m_i(U) = 1 - m_i(S) \quad (8)$$

Case B: $t_{1,i}^2 > T^2(\alpha)$

$$m_i(S) = 0 \quad (9)$$

$$m_i(NS) = [2\alpha - 1 + (1 - \alpha)\, Pr\{T^2 \leq t_{1,i}^2\}]/\alpha \quad (10)$$

$$m_i(U) = 1 - m_i(NS) \quad (11)$$

We have derived the formulas for the above belief function so that the following limiting conditions are satisfied:

(i) If $t_{1,i}^2$ decreases to zero, then $m_i(S)$ increases to 1

(ii) If $t_{1,i}^2$ increases to infinity, then $m_i(NS)$ increases to 1

(iii) The belief attributed to uncertainty, $m_i(U)$, is maximum at $T^2(\alpha)$

(iv) There is no jump in $m_i(U)$ at $T^2(\alpha)$

The percentage points of the T^2 distribution for any given level of significance and the cumulative probability for any given value of the T^2 random variable can be obtained, as described in the Appendix of Narasimhan et at. (1986).

In the belief function we have proposed, probabilities from the T^2 distribution are used to assign directly beliefs for the propositions S, NS, and U in case A, whereas in case B the probabilities are weighted appropriately to calculate the beliefs. Alternatively, one may directly use the probabilities from the T^2 distribution to obtain the beliefs in case B and weight the probabilities to obtain the beliefs in case A such that conditions (i) and (iv) are satisfied. One can also modify condition (ii) such that the belief for NS equals 1, if $t_{1,i}^2$ is greater than or equal to a finite number, and accordingly compute the beliefs for case B.

Dempster' rule for combining beliefs

The beliefs that are obtained from the different variables must be combined in order to decide if the process is in a steady state or not. For this purpose we make use of Dempster's rule of combination (Shafer, 1976). The overall belief for each proposition, which is obtained by using Dempster's rule, is given as follows:

$$m(S) = \prod_{i=1}^{p} [m_i(S) + m_i(U)] \quad (12)$$

$$m(NS) = \prod_{i=1}^{p} [m_i(NS) + m_i(U)] \quad (13)$$

$$m(U) = \prod_{i=1}^{p} [m_i(U)] \quad (14)$$

where p is the total number of variables. Equations 12, 13, and 14 give the unnormalized overall beliefs, which may be normalized so that the sum of the beliefs of all propositions equals unity. It should be noted that Dempster's rule assumes that the individual beliefs are obtained from independent knowledge sources (variables), which in our case is true by the assumption that $\underline{Q}$ is diagonal.

Based on the overall beliefs, we decide that the process is in a steady state if $m(S)$ is greater than $m(NS)$. Otherwise we decide that the process state has changed. Note that we do not have to normalize the beliefs or calculate $m(U)$ for this purpose.

Simulation Results and Discussion

Simulation studies were performed to compare the performance of the MTE with that of classical hypotheses-testing to detect process steady state changes. Since we have assumed that $\underline{Q}$ is constant, a multivariate composite statistical test described by Narasimhan et al. need not be applied. Instead, we can use the multivariate $T_{p,2N-2}^2$ test for equal but unknown covariance matrices (test 2A in Narasimhan et al., 1986).

The data that we need to simulate are the mean values and the sample variances in each period. Let δ_i be the change in the true value and q_{ii} the variance for variable i. The mean value of variable i in one period is generated by a random variable from $N(0, q_{ii}/N)$, and in the next period by a random variable from $N(\delta_i, q_{ii}/N)$, and so on alternately. The sample variance for each variable i in each period is generated by a random variable from the chi-square distribution with one degree of freedom, multiplied by q_{ii}/N. If δ_i is 0 for all i, then a steady state is simulated. Otherwise a change in process steady state, as specified by changes δ_i in each variable i, is simulated. In each simulation run, 10,000 simulation trials are made. In each simulation trial the test based on the MTE is applied to a pair of successive periods. The proportion of trials rejected in each simulation is computed. This proportion gives the probability of type I error

Table 1. Parameter Values Used in Simulation

Number of variables	2
Period size	60
Level of significance, MTE	0.10
Level of significance, $T^2_{p,2N-2}$ test	0.11
Variances of measurement errors	1.0, 2.0
Number of simulation trials	10,000

(resp., power) for a steady state (resp., nonsteady state) simulation. It should be noted that the probability of type I error and power of the MTE must be evaluated through simulation. However, for the $T^2_{p,2N-2}$ test these performance characteristics can be obtained analytically as described in the Appendix of Narasimhan et al. (1986).

The results of the simulation are presented in Tables 1 and 2. In Table 1, the values of the parameters that are the same for all simulation cases are listed. The levels of significance are chosen such that the probability of type I error for both methods is the same. Therefore, both methods are compared on the same basis.

The different simulation runs and their results are given in Table 2. For each simulation run, the magnitudes of the relative changes in the true values for the variables (δ_i/q_{ii}) are listed in columns 2 and 3. The proportion of trials rejected by the MTE is given in column 4, and the power of the $T^2_{p,2N-2}$ test obtained theoretically is given in column 5. In simulation run 1, the changes in the true values of all variables are equal to zero. Hence, for this run the proportion of trials rejected is equal to the probability of type I error for the MTE. Note that this value is the same as the level of significance of the $T^2_{p,2N-2}$ test given in Table 1, and therefore the probability of type I error is the same for both methods. The proportion of trials rejected for the other simulation cases represents the power of the MTE.

The error in the estimates of probability of type I error and power is ±0.01. Therefore, comparing columns 4 and 5 of Table 2, we can say that both methods give almost the same power. It should be noted that both methods give the same power when the relative changes in the variables are small (run 4) or large (run 2), or when only one variable undergoes a change in its true value (runs 5, 6 and 7). Similar results were obtained for the case of three variables, but these are not listed since they do not provide any new information. Based on these results, we can conclude that both methods give the same performance.

In some applications, it may also be important to know which variables caused a change in the state of the process to be detected. This is readily obtained in the case of the MTE by identifying the variables for which $m_i(NS)$ is greater than $m_i(S)$. In the case of $T^2_{p,2N-2}$ test, this information can be obtained only through additional testing. The MTE can also easily be modified to allow for prespecified tolerances for each variable within a steady state by including the tolerance in Eq. 3 when calculating the test statistic for each variable. However, one limitation of the MTE is that it is applicable only if the variables are independent. Another limitation is that the probability

Table 2. Simulation Results

Run No.	Relative Changes in Variables δ_1/q_{11}	δ_2/q_{22}	Proportion of Trials Rejected by MTE	Power of $T^2_{p,2N-2}$ Test
1	0.0	0.0	0.11	—
2	0.5	0.5	0.97	0.97
3	0.25	0.25	0.54	0.54
4	0.15	0.15	0.27	0.28
5	0.7	0.0	0.96	0.97
6	0.35	0.0	0.52	0.54
7	0.15	0.0	0.19	0.19

of type I error of the method is not readily given by the level of significance, as in the case of the $T^2_{p,2N-2}$ test. The user has to adjust iteratively the level of significance for the MTE in order to obtain a desired probability of type I error.

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Notation

$m_i(S)$ = belief for steady state proposition based on variable i
$m_i(NS)$ = belief for nonsteady state proposition based on variable i
$m_i(U)$ = belief for uncertainty based on variable i
N = number of measurements in each period
p = number of variables
$Pr\{.\}$ = probability of .
$\underline{Q}$ = covariance matrix of measurement errors
q_{ii} = ith diagonal element of $\underline{Q}$
$t^2_{1,i}$ = T^2 statistic for variable i, Eq. 3
$s_{k,i}$ = sample variance of variable i in period $\underline{k}$
$T^2_{a,b}$ = T^2 random variable with numerator degrees of freedom a and denominator degrees of freedom b
$T^2(\alpha)$ = upper T^2 percentile
v_{kj} = jth vector of random measurement errors in period k
$\underline{x}_{kj}$ = jth measurement vector in period k
$x_{kj,i}$ = jth measurement of variable i in period $\underline{k}$
x_k = vector of true values of variables in period k
$\bar{x}_{k,i}$ = mean value of measurements for variable i in period k

Greek letters

α = level of significance
Π_i = product over all i
δ_i = value change for variable i
Σ_i = sum over all i

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Errata

In the paper titled "Generalized Likelihood Ratio Method for Gross Error Identification" by S. Narasimhan and R. S. H. Mah (Sept. 1987), the Grant No. on p. 1520 should read CPE 8519182, not CPE 811 5161.